

## ALIGNMENT CORRESPONDENCES

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## 1. INTRODUCTION

Let  $X$  be a smooth, proper variety of dimension  $n$  over a field  $K$ . Let  $R$  be the ring  $K[[x_1, \dots, x_n]]$  and  $\mathfrak{m}$  its maximal ideal. Let  $I$  be an ideal of colength  $d$  in  $R$ . The space  $U(I)$  of subschemes of  $X$  isomorphic to  $\text{Spec}(R/I)$  has a natural embedding in the punctual Hilbert scheme  $\text{Hilb}^d(X)$  as a locally closed subvariety (Theorem 2.1). More generally, one can consider the space

$$U(I_1, \dots, I_r) = \{(a_1, \dots, a_r) \in U(I_1) \times \dots \times U(I_r) : \exists p \in X \text{ and}$$

$$\varphi : R \xrightarrow{\sim} \hat{\mathcal{O}}_{X,p} \text{ with } \varphi(I_1, \dots, I_r) = (a_1, \dots, a_r)\}$$

and its closure  $C(I_1, \dots, I_r)$  in the appropriate product of Hilbert schemes. If the ideals  $I_j$  are monomial, we will say that the space  $C(I_1, \dots, I_r)$  is an *alignment correspondences* with *interior*  $U(I_1, \dots, I_r)$ . The significance of the  $I_j$ 's being monomial is that in this case we can show that the space  $U(I_1, \dots, I_r)$  in most cases is an affine bundle on the flag bundle on  $X$  and in the remaining cases has an étale covering by such a space and give a classification of these spaces via measuring sequences (Theorem 3.1). While many interiors of alignment correspondences are naturally isomorphic while the alignment correspondences themselves may vary quite a bit. In particular, we show that in some there is a compactification of an isomorphism class of interiors of alignment correspondences dominating all alignment correspondences with that interior and sometimes there is not. Among the reasons for studying these spaces are for their applications to enumerative geometry,

to better understand Hilbert schemes, and because they are a fresh ground for finding something new.

Alignment correspondences and their interiors have appeared in several places in the mathematical literature previously. If  $X$  is a surface, given a Hilbert series  $T$  space  $\bar{Z}_T$  will be isomorphic to the fiber of  $C(I)$  over  $X$  for some ideal  $I$  which may or may not be monomial [4]. In the case that  $T$  is a series of  $m$  1's, the ideal  $I$  can be taken to be the ideal  $(x_1, x_2^m)$  and the space  $C(I)$  parametrizes all zero-dimensional subschemes of  $X$  supported at a point [1]. The spaces  $H(D)$  corresponding to an Enriques diagram  $D$  [5], are of the form  $U(I)$ , although  $I$  is not always a monomial ideal. The space  $U((x_1, \dots, x_{n-1}, x_n^{m+1}))$  is the space of curvilinear  $m$ -jets. These spaces were compactified by Semple [9]. These compactifications were described in modern language and coined the Semple bundles by Collino [2]. Some Semple bundles can be realized as alignment correspondences, but it is not certain whether all can be. The alignment correspondences  $X_m = C((x, y), (x, y^2), \dots, (x, y^m))$  were studied in [2]. These are also compactifications of the spaces of curvilinear jets, but do not coincide with the Semple bundles for  $m \geq 4$  [2]. Other familiar examples include flag bundles on the tangent bundle of  $X$  and the space the jet bundles  $U(x_1, \mathfrak{m}^n)$  parametrizing  $m - 1$ -jets in the usual sense.

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## 2. PRELIMINARIES

Let  $G(I_1, \dots, I_r)$  be the group of automorphisms of  $R$  fixing the ideals  $I_j$  for  $j$  from 1 to  $r$ . The fiber of  $U(I_1, \dots, I_r)$  over  $X$  can be identified with the quotient  $\text{Aut}(R)/G(I_1, \dots, I_r)$  via any isomorphism

$$\varphi : R \rightarrow \hat{\mathcal{O}}_{X,p}.$$

If  $G(I_1, \dots, I_r) \subset G(J_1, \dots, J_s)$ , this identification induces a map from  $U(I_1, \dots, I_r)$  to  $U(J_1, \dots, J_s)$ . This map may or may not extend to the boundary. We will say that such maps as well as their extensions and restrictions are *natural*. In partial compensation for the fact that these maps do not always extend, we can consider the space  $C(I_1, \dots, I_r, J_1, \dots, J_s)$  which maps to  $C(I_1, \dots, I_r)$  and  $C(J_1, \dots, J_s)$  by projection. We will say this the space obtained by *superimposing*  $C(I_1, \dots, I_r)$  and  $C(J_1, \dots, J_s)$ . Through the isomorphism  $\varphi$ , the

group  $\text{Aut}(R)$  acts on the fiber of  $C(I_1, \dots, I_r)$  over  $X$ . More generally if  $C(I_1, \dots, I_r)$  admits a natural map to  $C(J_1, \dots, J_s)$ , assuming without loss of generality that  $\varphi$  takes a point  $P \in U(J_1, \dots, J_s)$  to  $(J_1, \dots, J_s)$  then  $G(I_1, \dots, I_s)$  acts on the fiber in  $C(I_1, \dots, I_r)$  over  $P$  through  $\varphi$ .

**Theorem 2.1.** *Given a sequence of ideals  $I_1, \dots, I_r$  of finite colengths  $d_1, \dots, d_r$  respectively the space  $U(I_1, \dots, I_r)$ , as defined in the introduction, is a locally closed subset of the space  $H = \text{Hilb}^{d_1}(X) \times \dots \times \text{Hilb}^{d_r}(X)$ .*

**Proof:** From the product of Hilbert Chow morphisms

$$\varphi_1 \times \dots \times \varphi_r : H \rightarrow \text{Sym}^{d_1}(X) \times \dots \times \text{Sym}^{d_r}(X),$$

we see that the points of  $H$  corresponding to sequences of schemes all supported at the same point is a closed subvariety  $Y$ . The space  $Y$  is a fiber bundle over  $X$  containing  $U(I_1, \dots, I_r)$ . A fiber of  $U(I_1, \dots, I_r)$  over  $X$  is an  $\text{Aut}(R)$  orbit of  $Y$ . Hence, covering  $X$  by neighborhoods in which there is a continuous section of  $U(I_1, \dots, I_r)$  and a continuous action of  $\text{Aut}(R)$  restricting on the fibers over  $X$  to the action described above, we see that the space  $U(I_1, \dots, I_r)$  is locally closed in  $Y$  and therefore in  $H$ .  $\square$

### 3. MEASURING SEQUENCES

**Definition:** Let  $I_1, \dots, I_r$  be a sequence of monomial ideals of finite colength in  $R$ . For each integer  $i$  between 1 and  $n$ , let  $A_i$  be the ideal generated by images of  $x_i$  under automorphisms of  $R$  fixing each  $x_j$  for  $i \neq j$  and sending each  $I_k$  to itself. We will say that the sequence  $A_1, \dots, A_n$  is the *measuring sequence* of the sequence of ideals  $I_1, \dots, I_r$ .

For the remainder of this paper we will continue to use the notation of the above definition.

**Proposition 3.1.** *Each ideal  $A_i$  in measuring sequence as just defined is a monomial ideal.*

**Proof:** Since the  $I_j$ 's are monomial ideals, the group of automorphisms of  $R$  fixing  $x_j$  for  $i \neq j$  and sending the  $I_j$ 's to themselves is stable under conjugation by the automorphisms scaling the  $x_j$ 's. This together with the fact that the ideal  $(x_i)$  is contained in  $A_i$  make  $A_i$  fixed by automorphisms scaling the  $x_j$ 's. Hence  $A_i$  is a monomial ideal.  $\square$

The measuring sequence of a sequence of ideals is often easy to calculate. If the characteristic of  $K$  is 0 then we have

$$A_i = \{f \in R : (I_j : x_i) \subset (I_j : f) \ \forall \ 0 \leq j \leq r\}.$$

In a positive characteristic  $p$ , one must also take into consideration what we will call the exponent types of the monomials generating the ideals.

**Definition:** If the characteristic of  $K$  is a positive integer  $p$ , say that a monomial  $f$  has *exponent type*  $(a_0, \dots, a_m)$  if  $f$  can be written as the product  $f_0 f_1^p \dots f_m^{p^m}$  where the  $f_i$ 's are  $p^{\text{th}}$  power free monomials of weight  $a_i$  if the weights have been given to the variables or degree  $a_i$  if no weights have been specified. Otherwise, say that the exponent type of  $f$  is  $(wt(f))$  or, if no weights have been specified,  $(\deg(f))$ .

**Proposition 3.2.** *Given a vector*

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

*whose coordinates are non-negative integers write*

$$\alpha = \sum_{i=0}^m p^i v_i,$$

*where each  $v_i$  is a vector with minimal non-negative integral coordinates and the  $m$  is sufficiently large. If the characteristic of  $K$  is positive, let  $F$  be the Frobenius map. Let*

$$A^\alpha = A_1^{\alpha_1} \dots A_n^{\alpha_n}.$$

*Then for each*

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

*in  $I_i$ , we have the containment*

$$A(\alpha) = A^{v_0} F(A^{v_1}) \dots F^m(A^{v_m}) \subset I_i.$$

**Proof:** Suppose, by way of contradiction, that there is an exponent vector  $\alpha$  and a monomial  $f$ , such that  $x^\alpha \in I_k$ ,  $f \in A(\alpha)$ , and  $f \notin I_k$ . Then  $f$  can be expressed as the product  $f_0 \dots f_m^{p^m}$ , where each  $f_i$  is a monomial in  $A^{v_i}$ . Choose  $\alpha$  so that each  $f_i = x^{w_i} f'_i$  with the degree of  $x^{w_i}$  maximal and  $f'_i \in A^{v_i - w_i}$ . Let  $a$  be the largest integer such that  $v_i \neq w_i$ . Let  $j$  be an integer such that the  $i^{\text{th}}$  coordinate  $b$  of  $v_a - w_a \neq 0$ . Then there are monomials  $h_1, \dots, h_b \in A_j$  such that  $f'_i = h_1 \dots h_b h$  with  $h \in A^{v_a - w_a - b e_j}$ . Let  $g$  be the automorphism of  $R$  with  $g(x_j) = x_j + h_1$  and  $g(x_i) = x_i$  for  $i \neq j$ . Then by Proposition 3.1,  $g \in G(I_1, \dots, I_r)$ . The monomial  $x^\alpha (h_1/x_j)^{p^b}$  has non-zero coefficient in  $g(x^\alpha)$  and hence is in  $I_k$ . Let  $\beta$  be the exponent vector corresponding to this monomial. Then  $f$  is in  $A^\beta$ , but this contradicts the maximality of the degrees of the  $x^{w_i}$ 's.  $\square$

**Corollary 3.1.** *The natural map*

$$\varphi : U(A_1, \dots, A_n) \rightarrow U(I_j)$$

*is given by sending a point  $(a_1, \dots, a_n)$  to the sum ideals of the form  $a(\alpha)$  for a set of  $x^\alpha$ 's generating  $I_j$  and  $a(\alpha)$  given by replacing each  $A_i$  by  $a_i$  in  $A(\alpha)$  as in Proposition 3.2.*

**Proof:** The fact that the  $x^\alpha$ 's generate  $I_j$  ensures that the sum of the ideals of the form  $a(\alpha)$  contains  $\varphi(I_j)$ . The fact that the sum of these ideals is contained in  $\varphi(I_j)$  follows from Proposition 3.2.  $\square$

**Corollary 3.2.** *The group  $G(A_1, \dots, A_n)$  consists of all the automorphisms of  $R$  sending each  $x_i$  to an element of  $A_i$ .*

**Proof:** The group  $G(A_1, \dots, A_n)$  is contained in the set of automorphisms of  $R$  sending each  $x_i$  to an element of  $A_i$  because  $x_i \in A_i$ . By Proposition 3.2, it follows that all such automorphisms of  $R$  are contained in  $G(A_1, \dots, A_n)$ .  $\square$

**Lemma 3.1.** *There is a bijection between nested sequences*

$$\mathfrak{m}^2 \subsetneq B_1 \subsetneq \dots \subsetneq B_m = \mathfrak{m}$$

*of distinct monomial ideals such that  $G(A_1, \dots, A_n) \subset G(B_1, \dots, B_m)$ , with  $m$  is maximal and completions of the partial orderings on the  $x_i$ 's such that  $x_i \leq x_j$  exactly when  $x_i \in A_j$  to total orderings in which no new equivalences of variables are introduced.*

**Proof:** Given the set of  $B_i$ 's, let  $<$  be the (possibly non-strict) total ordering on the  $x_i$ 's such that  $x_i \leq x_j$  if  $x_j \in B_k$  implies  $x_i \in B_k$ . Conversely, given such a total ordering, there is a unique set of  $B_i$ 's from which it comes. The fact that the total ordering contains the partial ordering coming from the  $A_i$ 's ensures  $G(A_1, \dots, A_n) \subset G(B_1, \dots, B_m)$ . The fact that the  $<$  is completed to a total ordering makes  $m$  maximal.  $\square$

Henceforth we will let  $B_1, \dots, B_m$  denote a sequence of ideals as in Lemma 3.1 and  $B$  denote the space  $C(B_1, \dots, B_m)$ . Then  $B$  is a flag bundle on the tangent bundle of  $X$  that is maximal with the property that  $U(A_1, \dots, A_n)$  admits a natural map to it. Moreover, we will let  $G(B)$  denote  $G(B_1, \dots, B_m)$ ,  $C(I_1, \dots, I_r, B)$  denote  $C(I_1, \dots, I_r, B_1, \dots, B_m)$ , etc.

**Theorem 3.1.** 1. *The space  $U(A_1, \dots, A_n)$  can be expressed as an affine bundle over  $B$ .*

2. *The natural map*

$$U(A_1, \dots, A_n) \rightarrow U(I_1, \dots, I_r)$$

is an étale covering.

3. The space  $C(I_1, \dots, I_r, B)$  is a compactification of  $U(A_1, \dots, A_n)$  which is a fiber bundle over  $B$ .
- 4.

$$\sum_{i=1}^n \text{col}(A_i) = \dim(C(I_1, \dots, I_r)).$$

**Proof of Theorem 3.1(1)** Let  $W_1 \oplus \dots \oplus W_m$  be a decomposition of the vector space  $\mathfrak{m}/\mathfrak{m}^2$  into direct summands spanned by monomials such that  $B_i = W_1 \oplus \dots \oplus W_i$ . Let  $N$  be large enough so that each  $A_i$  contains  $\mathfrak{m}^N$ . For  $i$  from 1 to  $m$ , let  $J_i$  be the ideal generated over  $\mathfrak{m}^N$  by the monomials spanning  $W_i$ . Then  $G(J_1, \dots, J_m)$  is contained in  $G(A_1, \dots, A_n)$  and is expressible as the direct sum of the group of the degree preserving automorphisms sending the  $W_i$ 's to themselves with the group of automorphisms acting trivially on  $\mathfrak{m}/\mathfrak{m}^N$ . By Lemma 3.2, a set of coset representatives for  $G(A_1, \dots, A_n)/G(J_1, \dots, J_m)$  is given by the set of all automorphisms of  $R$  sending each  $x_i$  to the sum of  $x_i$ , a linear combination of monomials in  $A_i$  that are either of degree strictly between 1 and  $N$  and  $x_j$ 's in  $A_i$  that are inequivalent to  $x_i$ . A set of coset representatives for  $G(B)/G(J_1, \dots, J_m)$  is given by the set of automorphisms of  $R$  sending each  $x_i$  to  $x_i$  plus a linear combination of monomials of degree strictly between 1 and  $n$  and  $x_j$ 's strictly less than  $x_i$ . It follows that  $U(J_1, \dots, J_m)$  is an affine bundles over both  $U(A_1, \dots, A_n)$  and  $B$ . Therefore,  $U(A_1, \dots, A_n)$  is an affine bundle over  $B$  as well.  $\square$

**Lemma 3.2.** *Let  $W_1 \oplus \dots \oplus W_m$  be a decomposition of the vector space  $\mathfrak{m}/\mathfrak{m}^2$  into nontrivial direct summands with  $m$  maximal and each subspace  $W_1 \oplus \dots \oplus W_i$  fixed by some power of any element of  $G(I_1, \dots, I_r)$ . Then the group  $G$  of degree preserving automorphisms of  $R$  fixing the  $W_i$ 's is contained in  $G(I_1, \dots, I_r)$ .*

**Proof:** We can assume that  $K$  is a perfect field containing an element that is neither a root of unity nor 0, because if the statement of the lemma holds over a given field, then it also holds over any subfield. By this assumption, the  $W_i$ 's are uniquely determined up to permutation by maximality of  $m$  and spanned by monomials since they are invariant under scaling by some unit of  $K$  that is not a root of unity.

Let  $M$  be the set of monic monomials in  $R$ . Let the  $G$ -topology on  $M$  be the topology such that the closed sets are those with span fixed by the group  $G' = G(I_1, \dots, I_r) \cap G$ . Let  $B(f)$  denote the closure of a monomial  $f$  with respect to this topology. We will show that the spans

of the closed sets are fixed by  $G$  by showing that the  $G$ -topology is the same as the exponent topology defined as follows.

Let  $<$  be the partial ordering on exponent types such that

$$(a_0, \dots, a_m) \leq (m_0, \dots, m_b)$$

exactly when

$$\sum_{i=0}^k a_i p^i \leq \sum_{i=0}^k m_i p^i$$

for all  $k$ , with eventual equality, taking  $a_l$  (respectively  $m_l$ ) to be 0 if  $l > m$  (respectively  $l > b$ ). Thus only monomials of the same weight or degree are comparable.

Say that a monomial is an  $i$ -monomial if it is a product of monomials in  $W_i$ . Let the *exponent topology* on  $M$  be the topology on  $M$  such that a set is closed if and only if whenever it contains a given  $i$ -monomial, then it contains all other  $i$ -monomials of lesser or equal exponent type.

Let  $M_i(d)$  be the set of  $i$ -monomials of degree  $d$ . Expressing  $M$  as the product

$$\prod_{i,d} M_i(d)$$

both topologies are box topologies. In the case of the  $G$ -topology, this follows from the fact that there is no cancellation of terms when one multiplies polynomials in the spans of distinct  $M_i(d)$ 's. Thus we have reduced the problem to showing that on each  $M_i(d)$  the  $G$ -topology and exponent topology are the same.

In one direction, it is clear that the  $G$ -topology is as fine as the  $p$ -topology. Suppose, by way of contradiction, that there are an  $l$  and a  $d$  such that the two topologies on  $M_l(d)$  are different, with  $d$  chosen to be minimal. Let  $B(a_1, \dots, a_t)$  denote the set of  $l$ -monomials of exponent type  $(a_1, \dots, a_t)$  and smaller types. If  $(a_0, \dots, a_t)$  is not a possible exponent type, we will use the convention that  $B(a_0, \dots, a_t)$  is empty. Then there is an  $l$ -monomial  $f$  of exponent type  $(a_0, \dots, a_t)$  such that  $B(f)$  is properly contained in  $B(a_0, \dots, a_t)$ . Choose  $f$  to have minimal exponent type. Note that we cannot have  $a_0 = 0$  because then the  $p^{\text{th}}$  root of  $f$  would give a counterexample of smaller degree. Relabelling the  $x_k$ 's if necessary, assume further that  $f$  can be expressed by

$$f = x_1^{e_1} \dots x_b^{e_b} h^p$$

with the degree of  $h$  and then the  $e_k$ 's maximal, in ascending order. Let  $\alpha$  be the sequence

$$a_0 - p, a_1 - p + 1, \dots, a_{c-1} - p + 1, a_c + 1, a_{c+1}, \dots$$

where  $c$  is the smallest integer with  $a_c \neq n(p-1)$ . Then  $B(\alpha)$  is the set of monomials in  $M_l(d)$  of exponent type smaller than  $(a_0, \dots, a_t)$  not beginning with  $a_0$ . Moreover, assume that  $f$  was chosen to give a vector space  $Z \subset M_l(1)$  of maximal dimension such that we have the containment

$$x_1^{e_1} \dots x_{b-1}^{e_{b-1}} Z^{e_b} h^p \subset B(f) \cup B(\alpha).$$

Given  $g \in G$  let  $i_k$  and  $x'_k$  be such that  $x'_k$  differs from  $g(x_k)$  by a linear combination of the  $x'_l$ 's for  $l < k$  and such that the coefficient of  $x_{i_l}$  in  $x'_k$  is 0 if  $l < k$  and non-zero if  $l = k$ .

By maximality of  $e_1$ , expanding  $g(x_1^{e_1} \dots x_b^{e_b})$  out in terms of the  $x'_k$ 's we see that modulo  $B(a_0 - p, 1)$  the largest power of  $x'_1$  occurring is  $e_1$  and hence that modulo  $B(a_0 - p, 1)$ , we can express  $g(x_1^{e_1} \dots x_b^{e_b})$  as the product of  $(x'_1)^{e_1}$  and a polynomial in the  $x'_k$ 's for  $k > 2$ . Continuing in this way, we see that we have

$$g(x_1^{e_1} \dots x_{b-1}^{e_{b-1}}) \equiv (x'_1)^{e_1} \dots (x'_b)^{e_b} \pmod{B(a_0 - p, 1)}.$$

Since  $g$  is invertible, there must be an  $h'$  of exponent type  $(a_1, \dots, a_t)$  such that the coefficient of  $h'$  in  $g(h)$  is non-zero. Any monomial with a non-zero coefficient in

$$x_1^{e_1} \dots x_{b-1}^{e_{b-1}} g(Z)^{e_b} (h')^p$$

is in  $B(f) \cup B(\alpha)$ . This is because modulo  $B(\alpha)$  there is no cancellation of terms when we multiply  $g(x_1^{e_1} \dots x_{b-1}^{e_{b-1}} Z^{e_b})$  with  $g(h^p)$ . By maximality of the dimension of  $Z$ , the monomial span of  $g(Z)$  must be of the same dimension as  $Z$ . Therefore,  $Z$  is fixed by some power of any element of  $G$  since there are only finitely many vector spaces generated by elements of  $M_l(1)$  with the same dimension as  $Z$ . Hence by minimality of  $W_l$  we have

$$Z = W_l.$$

By minimality of  $d$ , we have

$$B(x_1^{e_1} \dots x_{b-1}^{e_{b-1}} h^p) = B(a_0 - e_b, a_1, \dots, a_t).$$

Thus, the  $G$ -closure of the terms in  $x_1^{e_1} \dots x_{b-1}^{e_{b-1}} W_l^b h^p$  is  $B(a_0, \dots, a_t)$ . Therefore we have

$$B(f) \equiv B(a_0, \dots, a_t) \pmod{B(\alpha)}.$$

If  $a_0 \geq p$  or  $p = 0$ , we have reached a contradiction since then  $B(\alpha)$  is empty.

It remains to show that  $B(\alpha)$  is contained in  $B(f)$ . By minimality of  $(a_0, \dots, a_t)$ , it suffices to show that there is a single element of exponent type  $(\alpha)$  in  $B(f)$ . Given any  $l$ -monomial  $f_0$  of exponent type  $(a_0)$ , it follows from what we have just shown that  $f_0 B(a_1, \dots, a_t)^p$  is contained



in  $B(f)$ . Thus we need only show that  $B(f_0)$  contains a monomial of exponent type  $(a_0 - p, 1)$ . We will use

$$f_0 = (x_1 \dots x_{i_{b-1}})^{p-1} x_b^{e_b}$$

where the  $x_k$ 's are as before, noting that for  $k < b$  we have  $e_k = p - 1$ .

Let  $Z'$  be the maximal vector space such that the monomial span of

$$(x_1 \dots x_{i_{b-1}})^{p-1} (Z')^{e_b}$$

is contained in the span of  $B(f_0) \cup B(a_0 - 2p, 2)$ . Without loss of generality, assume that the  $x_i$ 's were chosen to maximize the dimension of  $Z'$ . Suppose, that there is an element  $g \in G'$  such that the monomial span of  $g(Z')$  has dimension greater than that of  $Z'$ . Expanding  $g((x_1 \dots x_{b-1})^{p-1} (Z')^{e_b})$  in the  $x'_k$ 's modulo  $B(a_0 - 2p, 2)$ , if there is a term of exponent type  $(a_0 - p, 1)$ , then we are done. Otherwise, we have

$$g((x_1 \dots x_{b-1})^{p-1} (Z')^{e_b}) \equiv (x'_1 \dots x'_{i_{b-1}})^{p-1} g(Z')^{e_b} \pmod{B(a_0 - 2p, 2)}.$$

Then since

$$(x'_1 \dots x'_{i_{b-1}})^{p-1} g(Z')^{e_b}$$

is also contained in  $B(f_0) \cup B(a_0 - 2p, 2)$ , by maximality of the dimension of  $Z'$ , any element of  $G'$  preserves the dimension of the monomial span of  $Z'$  and hence a power of any element of  $G$  fixes  $Z'$ . Therefore we see  $Z' = W_l$ .

Since there is a term of  $(x'_1 \dots x'_{i_{b-1}})^{p-1} (W_l)^{e_b}$  of exponent type  $(a_0 - p, 1)$ , there is a term of exponent type  $(a_0 - p, 1)$  in  $B(f_0)$ . Hence we have arrived at a contradiction. It follows that the exponent topology and the  $G$ -topology are the same. Therefore since the action of any element of  $G$  is continuous with respect to the exponent topology, it is continuous with respect to the  $G$ -topology and hence  $G$  must be contained in  $G(I_1, \dots, I_r)$ .  $\square$

**Proof of Theorem 3.1(2):** Throughout this proof we use the notation and intermediate steps from Lemma 3.2. Let  $B_i$  be the ideal in  $R$  consisting of representatives of elements of  $W_1 \oplus \dots \oplus W_i$ . Assuming again that  $K$  contains a unit of infinite order, the  $W_i$ 's are uniquely determined up to permutation. Thus any element  $g \in G(I_1, \dots, I_r)$ , is expressible as a product  $g_1 g_2$  where  $g_1$  preserves the ideals  $B_i$  and  $g_2$  is a permutation of the  $x_i$ 's sending each  $B_i$  to  $g(B_i)$ .

We will first show that  $g_2 \in G(I_1, \dots, I_r)$ . Suppose that there is a monomial  $f$  of smallest degree such that  $g_1(f)$  has 0 coefficient in every element of  $Gg(f)$ . Write  $f = f_1 f_2$  where  $f_1$  is an  $i$ -monomial for some  $i$  and  $f_2$  is a product of  $j$ -monomials for  $j < i$ . Let  $\sigma$  be the corresponding element of  $S_n$ . Then the terms of elements of  $Gg(f_2)$  are

products of  $W_{\sigma(j)}$  monomials for  $j < i$ . Thus given  $g_3 \in Gg$ , there is no cancellation of terms when multiplying the part of  $g_3(f_1)$  spanned by  $\sigma(i)$  monomials with  $g_3(f_2)$ . Hence, by minimality of  $f$ , either  $f_1$  or  $f_2$  must be a unit. Choose  $i$  so that  $f_2$  is a unit. Then since  $g$  is invertible, there must be a non-zero term in  $g(f)$  that is a  $\sigma(i)$ -monomial of the same exponent type as  $f$ . Thus by the proof of Lemma 3.2, all  $\sigma(i)$  monomials of this exponent type occur in some element of  $Gg(f)$ , including  $g_2(f)$ . Thus we have arrived at a contradiction and it follows that  $g_2 \in G(I_1, \dots, I_r)$ .

Next we will show that  $g_1 \in G(A_1, \dots, A_n)$ . The element  $g_1$  has a unique expression as the product of an element  $g_4 \in G$  and an element  $g_5$  such that for each  $f \in B_i$  the difference  $f - g_5(f) \in B_{i-1}$ . By Lemma 3.2 together with Lemma 3.2, it follows that  $g_4 \in G(A_1, \dots, A_n)$ .

One can see that  $g_5$  is an element of  $G(A_1, \dots, A_n)$  as follows. Let  $g' \in g_5 G(A_1, \dots, A_n)$  with

$$g'(x_k) = x_k + f_k$$

and the smallest degree terms in the  $f_k$ 's of maximal degree. Suppose that some  $f_k$  has its lowest piece  $h_k$  in  $A_k$ . Then composing  $g'$  with the element of  $G(A_1, \dots, A_n)$  sending  $x_k$  to  $x_k - h_k$  and fixing the other  $x_i$ 's we get a new automorphism contradicting the maximality of the degree of the lowest graded piece of  $f_k$ .

Suppose that  $g'$  is not the identity map. Choose  $k$  so that the degree of  $h_k$  is minimal and if this degree is 1, then  $x_k$  is minimal with respect to the ordering  $<$  associated to the  $B_i$ 's as in the proof of Theorem 3.1 (1). Since  $h_k \notin I_k$ , there must be a monomial  $f \in I_j$  for some  $j$  such that the automorphism  $g_6$  sending  $x_k$  to  $x_k + h_k$  and fixing the other  $x_i$ 's does not send  $f$  to an element of  $I_j$ . However, the terms of  $g_6(f)$  other than  $f$  are the same as those in  $g'(f)$  of the corresponding degree having the same power of  $x_k$ . Thus each monomial in  $g_6(f)$  and hence  $g_6(f)$  itself must be in  $I_j$ . Thus  $g'$  is the identity map and so  $g_5 \in G(A_1, \dots, A_n)$ .

Thus  $G(A_1, \dots, A_n)$  is the quotient of  $G(I_1, \dots, I_r)$  by the group of permutations of the  $x_i$ 's fixing the  $I_j$ 's modulo those fixing the  $A_j$ 's. It follows further that  $U(A_1, \dots, A_n)$  is an étale covering of  $U(I_1, \dots, I_r)$  of degree equal to the order of that group.  $\square$

**Lemma 3.3.** *Let  $\mathcal{I}$  be the set of pairs  $(i, v)$  such that  $x^v$  is in the complement of  $A_i$  and if  $|v| = 1$ , then  $x^v < x_i$  as in the proof of Theorem 3.1 (1). Then a set of right coset representatives for  $G(B)$*

over  $G(A_1, \dots, A_n)$  is given by the set  $\mathcal{S}$  of automorphisms  $g$  with

$$g(x_i) = x_i + \sum_{(i,v) \in \mathcal{I}} a_{i,v} x^v$$

for each  $i$ .

**Proof:** Let us first show that if  $g \in \mathcal{S}$  and  $h \in G(A_1, \dots, A_n)$  are automorphisms such that  $g \circ h \in \mathcal{S}$ , then  $h$  is the identity map. Suppose that  $h$  is not the identity map. Let  $k$  be such that the lowest graded piece  $h_k$  of  $h(x_i) - x_i$  is minimal. The ordering  $<$  can be extended to a partial ordering on monomials with the relations  $f_1 f_3 \leq f_2 f_4$  if  $f_1 \leq f_2$  and  $f_3 \leq f_4$ . Then for each monomial  $f$ , the terms of  $h(f) - f$  are all either of higher degree than  $f$  or of the same degree and strictly greater than  $f$ . The strictness comes from the consequence of Lemma 3.2 that  $G$  is contained in  $G(A_1, \dots, A_n)$ . Hence the minimal terms of  $h_k$  remains terms of  $g \circ h(x_k)$ . Thus by Lemma 3.2,  $g \circ h \notin \mathcal{S}$ . Thus we have arrived at a contradiction and it follows that  $h$  is the identity map. Moreover, by the proof of Theorem 3.1(1), the space  $\mathcal{S}$  is an affine space of the same dimension as  $G(B)/G(A_1, \dots, A_n)$  and so must be a full set of coset representatives.  $\square$

**Proof of Theorem 3.1(3):** By Lemma 3.3 the dimension of the fiber of  $U(A'_1, \dots, A'_n)$  over  $U(B)$  is the cardinality of  $\mathcal{I}$  in accordance with the theorem. Thus it remains to check that  $B$  has the dimension predicted by the theorem. If  $m = 1$  then the measuring sequence for the sequence  $B_1, \dots, B_m$  is a sequence of  $n$  copies of the ideal  $\mathfrak{m}$ . Thus the sum of the colengths of measuring ideals and the dimension of the space  $C(\mathfrak{m})$  are equal to the dimension of  $X$ . Suppose that the space  $C(B_2, \dots, B_m)$  has the expected codimension. The measuring sequences of  $B_1, \dots, B_m$  and  $B_2, \dots, B_m$  differ by  $\text{cod}(B_1)$  ideals. The differing ideals will be  $B_1$  and  $B_2$  respectively. Hence the difference of the sums of the colengths of ideals in the measuring sequences is

$$\text{cod}(B_1)(\text{cod}(B_1) - \text{cod}(B_2)).$$

It remains to be shown that this is the dimension of the fibers of  $C(B_1, \dots, B_m)$  over  $C(B_2, \dots, B_m)$ . The points of  $C(B_2, \dots, B_m)$  correspond to flags  $V_m \subset \dots \subset V_2$  in the tangent bundle of  $X$  where  $V_i$  is a subspace of a tangent space of a point of dimension equal to the codimension of  $B_i$ . The fiber over this flag consist of flags containing these subspaces, but with an extra subspace  $V_1$  containing  $V_2$  of dimension equal to the codimension of  $B_1$ . The dimension of the Schubert cell parameterizing possible  $V_1$ 's is  $(\dim(V_1) - \dim(V_2))(\dim(B_1))$  since

$V_1/V_2$  is spanned by  $\dim(V_1) - \dim(V_2)$  vectors taken from the quotient of the tangent space by  $V_2$ .  $\square$

**Proof of Theorem 3.1(4):** Given any point  $P \in B$  over a point  $p \in X$ , one can find functions  $f_0, \dots, f_n$  on  $X$  defined in a neighborhood of  $p$  such that there is an isomorphism

$$\varphi_P : \hat{R} \rightarrow \hat{\mathcal{O}}_{X,p}$$

with the following two properties.

1. There are constants  $a_{ij}$  such that we can write

$$\varphi_P(x_i) = \sum_{j=0}^n a_{ij} f_j$$

and if  $x_j \prec x_i$  then  $a_{ij} = \delta_{ij}$ .

2. The restriction of  $\varphi_P$  to  $\Lambda_2$  lies above  $P$ .

There is an open neighborhood  $U$  of  $P$  such that for each  $P' \in U$  there is a uniquely determined isomorphism  $\varphi_{P'}$  satisfying the above two properties with  $P$  replaced by  $P'$ . The isomorphisms  $\varphi_{P'}$ , being uniquely determined, vary continuously in  $U$ . Thus about each point of  $B$  we can find a local trivialization of  $C(I_1, \dots, I_r, B)$ .  $\square$

#### 4. UNIVERSALITY

**Definition:** If an alignment correspondence admits a natural map to a flag bundle on the tangent bundle of  $X$  over which its interior is an affine bundle, we will say that the alignment correspondence is *directed*. For example the alignment correspondence  $C(I_1, \dots, I_r, B)$  is directed.

By Theorem 3.1, every alignment correspondence is dominated by a one that is directed. Thus for most applications, one need only understand directed alignment correspondences. Fibers of directed alignment correspondences can be understood by their embedding in a Grassmannian of subspaces of an appropriate quotient of ideals.

##### Example 4.1.

Consider the fiber  $F$  of  $C((x, y^4), (x, y^2))$  over the space  $C((x, y^2))$ . By Lemma 3.3, the  $G((x, y^2))$  orbit of  $(x, y^4)$  is the  $\mathbb{A}^2$  of ideals of the form

$$(x + ay^2 + by^3, y^4).$$

As a subspace of the quotient  $(x, y^2)/(x, y^2)^2$ , a basis for such an ideal is given by

$$\{xy + ay^3, x + ay^2 + by^3\}.$$

Expanding

$$(xy + ay^3) \wedge (x + ay^2 + by^3)$$

with respect to the basis

$$\{y^3 \wedge xy, y^3 \wedge y^2, y^3 \wedge x, xy \wedge y^2, xy \wedge x, y^2 \wedge x\}$$

of  $\wedge^2 V_4$ , we get a map from  $F$  to  $\mathbb{P}_{x_0, \dots, x_5}^5$  given by

$$(x + ay^2 + by^3, y^4) \rightarrow (-b, a^2, a, a, 1, 0).$$

The closure of the image is cut out by

$$x_2 - x_3 = x_5 = x_1 x_4 - x_2^2 = 0.$$

Thus,  $F$  is a cone over a conic. The boundary parameterizes the  $\mathbb{P}^1$  of ideals of the form

$$(\alpha xy + \beta y^2, x^2, xy^2, y^3).$$

The cone point corresponds to the ideal

$$(x^2, xy, y^3).$$

□

**Theorem 4.1.** *Let  $x_i$  have weight  $e_i$  and let  $g$  be as in Lemma 3.3. Considering the  $a_{i,v}$ 's as coordinates for the fiber  $F$  of  $U(A_1, \dots, A_n)$  over  $B$ , give  $a_{i,v}$  weight  $e_i - v$ , making each  $g(x_i)$  homogenous. Then there are homogeneous coordinate functions embedding  $F$  in projective space in such a way that the closure  $\bar{F}$  is the fiber of  $C(A_1, \dots, A_n)$  over  $B$ . Moreover, if the weights of the coordinates on  $F$  are independent, the normalization of  $\bar{F}$  is a toric variety.*

**Proof:** There is a natural embedding of  $\bar{F}$  in the product a product of  $n$  Grassmanians. In particular, let  $V_i$  be the quotient of the union of the ideals in the  $G(B)$  orbit of  $A_i$  by the intersection of the  $G(B)$  orbit of  $A_i$ . Then the projection of  $C(A_1, \dots, A_n)$  to  $C(A_i)$  gives rise to a map from  $\bar{F}$  to the Grassmanian of subspaces of  $V_i$  of the appropriate dimension. In particular, the point of  $F$  with coordinates  $a_{i,v}$  is sent to the point corresponding to  $g(A_i)$  viewed as a subspace of  $V_i$ . These Grassmanians can then be embedded in projective space via homogeneous Plücker coordinates. Since  $g$  preserves homogeneity, with respect to these coordinates, the coordinate functions will be homogeneous. Mapping the product of projective spaces into a single projective space via the Segre embedding, the coordinate functions for  $F$  are products of homogeneous coordinate functions and hence homogeneous. If the weights of the coordinate functions are independent,

then the coordinate functions can only be monomials. Hence the normalization of  $\bar{F}$  is a toric variety. The action of scaling the  $x_i$ 's gives the action of an open dense torus.  $\square$

**Definition:** Say that  $x_i$  is *equivalent* to  $x_j$  if  $x_i \in A_j$  and  $x_j \in A_i$ . Say that two monomials are *equivalent* if they are of the same degree in the variables in each equivalence class. Say that variables  $a_{i,v}$  and  $a_{j,w}$  are *equivalent* if  $x_i$  is equivalent to  $x_j$  and  $x^v$  is equivalent to  $x^w$ .

**Theorem 4.2.** *Suppose that there are two inequivalent coordinates for the fiber  $F$  of  $U(A_1, \dots, A_n)$  over  $B$ . Then there is no variety  $Y$  of finite type admitting a dominant map to every alignment correspondence with measuring sequence  $A_1, \dots, A_n$  such that these maps commute with all restrictions of the natural maps between alignment correspondences.*

**Proof:** Fixing the measuring sequence  $A_1, \dots, A_n$ , it is enough to show that the theorem holds for one choice of  $B$  because if  $X$  exists for one choice of  $B$ , then  $X$  is universal for all choices of  $B$ . Let  $\prec$  be the ordering associated to  $B$  as in Lemma 3.1. Without loss of generality, assume that the indices of the  $x_k$ 's are such that  $\prec$  contains the total ordering on the  $x_k$ 's inherited from the total ordering on the indices. If there are inequivalent coordinates for one ordering  $\prec$ , then there are inequivalent coordinates for all orderings.

We first show that for some choice of  $B$  there are two coordinates with indices given by one of the entries of the following table. Suppose that these is a pair of inequivalent coordinates  $a_{i,e_j}$  and  $a_{k,e_l}$  chosen to minimize the larger of  $i - j$  and  $k - l$ . This difference must be 1 and the indices for the pair of coordinates must be as in the first or second row of the first column of the table. Suppose that there are no two such inequivalent coordinates. Then there is a coordinate  $a_{i,e_j+e_k}$  with  $i$  minimal and  $j$  and  $k$  maximal. Let  $\prec$  be chosen so that  $i = 1$  and  $e_j + e_k$  is either equal to  $2e_n$  or  $e_n + e_{n-1}$ .

In the former case, if  $n \neq 2$ , one can chose  $B$  so there is an inequivalent coordinate corresponding to the second entry in the first column of the third, fourth, or fifth row. If  $n = 2$ , then there will be a second coordinate corresponding to the entry in the sixth row.

In the latter case, if characteristic of  $K$  is not 2, then one can choose  $B$  so that the indices of a pair of inequivalent coordinates are given by the entries in the first column of the seventh row. In characteristic 2, for a suitable choice of  $\prec$ , there will be a pair of inequivalent coordinates corresponding to the entries in the first column of one of the last three rows. Thus, one can always choose  $B$  so that there is a pair of inequivalent coordinates with indices given by one of the rows of the

$\mathcal{S}$	$I/J_2$	$J_1/I$
$i, e_{i-1}$ $j, e_{j-1}$	$x_i^m x_{j-1}, \dots, x_i x_{i-1}^{m-1} x_{j-1},$ $x_{i-1}^m x_j$	$x_{i-1}^m x_{j-1}$
$i, e_{i-1}$ $i+1, e_i$	$x_i^{m+1}, \dots, x_{i-1}^{m-1} x_i^2, x_{i-1}^{m+1}$ $x_{i-1}^m x_{i+1}$	$x_{i-1}^m x_i$
$1, 2e_n$ $1, e_{n-1} + e_n$	$x_1^m x_{n-1}, \dots, x_1 x_n^{2m-2} x_{n-1},$ $x_1 x_n^{2m-1}$	$x_n^{2m} x_{n-1}$
$1, 2e_n$ $2, 2e_n$	$x_1^m, \dots, x_1 x_n^{2m-2},$ $x_2 x_n^{2m-2}$	$x_n^{2m}$
$1, 2e_n$ $i, e_{i-1}$	$x_1^m x_{i-1}, \dots, x_1 x_{i-1} x_n^{2m-2},$ $x_i x_n^{2m}$	$x_{i-1} x_n^{2m}$
$1, 2e_n$ $1, 3e_n$	$x_1^m, \dots, x_1 x_n^{2m-2},$ $x_1 x_n^{2m-3}$	$x_n^{2m}$
$1, e_{n-1} + e_n$ $n, e_{n-1}$	$x_1^m, \dots, x_1 x_{n-1}^{2m-2},$ $x_{n-1}^{2m-1} x_n$	$x_{n-1}^{2m}$
$1, e_{n-1} + e_n$ $i, e_{i-1}$	$x_1^m x_{i-1} x_{n-1}^{N-1} x_n^{N-1}, \dots, x_{i-1} x_{n-1}^N x_n^N,$ $x_1 x_i x_{n-1}^{N-1} x_n^{N-1}$	$x_{i-1} x_{n-1}^N x_n^N$
$1, e_{n-1} + e_n$ $2, e_{n-1} + e_n$	$x_1^m x_{n-1}^{N-m} x_n^{N-m}, \dots, x_1 x_{n-1}^{N-1} x_n^{N-1},$ $x_2 x_{n-1}^{N-1} x_n^{N-1}$	$x_{n-1}^N x_n^N$

first column of the table. Without loss of generality, assume that  $i < j$  in the first row of the table. In the last two rows  $N$  is an integer larger than  $m$  that is 1 less than a power of 2.

Fix a row of the table. Henceforth, we will be looking only at the entries in that row. Let  $J_2$  be the ideal generated by monomials  $f$  with weight at least that of all of the monomials in the second and third columns with respect to the weighting in which  $x_k$  has weight 1 for  $x_a \prec x_k$  (not excluding  $x_k \prec x_a$ ) and weight 2 otherwise, and either the weight of  $f$  or the weighted exponent type is strictly greater than those of each these monomials for some  $a$ . Let  $I_m$  be the ideal generated over  $J_2$  by the monomials of the second column and equivalent monomials. Then  $J_2$  is the intersection of the ideals in the  $G(B)$  orbit of  $I_m$  and

these monomials are a basis for  $I_m/J_2$ . Moreover, these monomials together with the monomials equivalent to the one in the third column are a basis for the union  $J_1$  of the ideals in the  $G(B)$  orbit of  $I_m$  over  $J_2$ . Let  $\mathcal{S}$  the set of elements in the first column. Let  $L$  be the lattice generated by the vectors  $e_i - v$  for  $i, v \in \mathcal{S}$ . Let  $\mathcal{A}$  denote the set of automorphisms  $g$  of  $R$  with

$$g(x_i) = x_i + \sum_{(i,\alpha) \in \mathcal{S}} a_{i,\alpha} x^\alpha$$

for each  $i$  from 1 to  $n$ . Let  $M$  be the subspace of  $J_1/J_2$  generated by the monomials in the second and third columns. Then the differences of the weights of these monomials lie in  $L$ .

The closure of the  $\mathcal{A}$  orbit of  $I_m$  in the Grassmanian of hyper-planes in  $J_1/J_2$  is a cone over a rational normal curve of degree  $m$ . The  $\mathcal{A}$  orbit of  $(I_2, \dots, I_m)$  in the appropriate product of Grassmanians. The boundary of the image is a chain of  $(m-1)$   $\mathbb{P}^1$ 's.

Suppose there is a variety  $Y'$  with a dominant maps  $f_m$  to the fiber of  $C(I_2, \dots, I_m, B)$  over  $B$  for all  $m$  such that these maps commute with the natural projection maps. If  $Y$  exists, then its fiber over a point in  $B$  is such a variety. Then the image of the boundary of the fiber of  $C(I_2, \dots, I_m, B)$  under  $f_m$  is independent of  $m$  and has at least  $m-1$  components. Thus  $Y'$  cannot be of finite type. Thus  $Y$  cannot be of finite type.  $\square$

**Theorem 4.3.** *If the measuring sequence  $A_1, \dots, A_n$  does not satisfy the hypotheses of Theorem 4.2 then the coordinates for the fiber  $F$  of  $U(A_1, \dots, A_n, B)$  over  $B$  are all coordinates  $a_{i,v}$  in an equivalence class satisfying one of the following:*

1.  $v = e_j$ ,
2.  $v = e_j + e_k$  with  $x_j$  and  $x_k$  equivalent.
3.  $v = e_j + e_k$  with  $x_j$  and  $x_k$  equivalent and  $j \neq k$
4.  $v = e_j + e_k$  with  $x_k$  in a unique equivalence class and  $j \neq k$ .

*The last two cases happen only in characteristic 2. Let  $m$  be the number of elements in the equivalence class of  $x_i$  and  $l$  the number in the equivalence class of  $x_j$  for any coordinate  $a_{i,v}$ . Then if  $m$  or  $l = 1$  in the first or fourth case,  $l = 1$  in the second, or  $l = 2$  in the third, then given a sequence  $(I_1, \dots, I_r)$  with measuring sequence  $(A_1, \dots, A_n)$ , the space  $C(A_1, \dots, A_n, B)$  dominates  $C(I_1, \dots, I_r)$  via the natural map.*

**Proof:** The fact that these are the only cases follows from the proof of Theorem 4.2. Let  $G$  be the group of linear automorphism of  $R$  fixing the subspace  $V_1$  of  $\mathfrak{m}/\mathfrak{m}^2$  spanned by the variables equivalent to  $x_i$ , fixing the subspace  $V_2$  spanned by variables equivalent to  $x_j$ , and fixing all



all variables. In the first and last case,  $F$  can be identified with the unipotent radical  $U_{m,l}$  of the parabolic subgroup  $P_{m,l}$  of  $GL(V_1 \oplus V_2)$  fixing  $V_1$ . The group  $G$  can be identified with the Levi subgroup of  $P_{m,l}$  acting by conjugation. In the second case  $F$  can be identified with the space of  $m$ -tuples of quadratic forms in  $l$  variables. An element of  $G$  viewed as an element  $(g_1, g_2) \in GL(V_1) \times GL(V_2)$  acts by the composition of the action of  $g_1$  by change of variables and the action of  $g_2$  by matrix multiplication. The third case is similar to the second, except that the space of  $m$ -tuples of quadratic forms is taken modulo the image of the Frobenius on the space of linear forms.

If  $m$  and  $l$  are as in the hypotheses of the theorem then  $G$  is the group of all linear transformations of the affine space  $F$ . In the third case, this is less straightforward, but using the fact that the characteristic is 2,  $GL(V_2)$  acts on  $F$  by the cofactor matrix of the matrix giving the action on  $V_2$ . The fiber of  $C(A_1, \dots, A_n, B)$  over  $B$  is a projective space in which the only fixed subvariety under  $G$  is the boundary. Thus it follows that it is the universal  $G$ -equivariant compactification of  $F$  and hence that  $C(A_1, \dots, A_n, B)$  is universal.  $\square$

In the four types of measuring sequences listed in Theorem 4.3, the coset representatives for  $G(B)/G(A_1, \dots, A_n)$  given in Lemma 3.3 form an algebraic group. Thus equivariant compactifications of these groups arise naturally as fibers of alignment correspondences with these measuring sequences over  $B$ .

## REFERENCES

- [1] Briançon, J. *Description de  $\text{Hilb}^n \mathbb{C}\{x, y\}$* . Inventiones math. **41**, 45–89 (1977).
- [2] Collino, A. *Evidence for a conjecture of Ellingsrud and Strømme on the Chow ring of  $\text{Hilb}_d(\mathbb{P}^2)$* . Illinois Journal of Mathematics **32**, 171–210 (1988).
- [3] Fulton, W. *Introduction to Toric Varieties*, Princeton University Press, Princeton, 1993.
- [4] Iarrobino, A. *Punctual Hilbert schemes* Mem. Amer. Math. Soc. **10** no. 188 (1977).
- [5] Kleiman, S., R. Piene *Enumerating singular curves on surfaces*, Algebraic Geometry: Hirzebruch 70 (Warsaw 1998), 209–238, Contemp. Math., 241, Amer. Math. Soc, Providence, RI, 1999.
- [6] Nakajima, H. *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, **18**, Amer. Math. Soc., Providence, RI, 1999.
- [7] Russell, H. *The Enumeration of Plane Curves with Singularities Corresponding to Monomial Ideals* (preprint)
- [8] Russell, H. *Hilbert Schemes and Monomial Ideals*, thesis, Harvard University, June 1999.
- [9] Semple, J. *Some investigations in the geometry of curves and surface elements*. Proc. London Math. Soc. (3), vol. 4 (1954), pp 24–49.
- [10] Vakil, R. *A beginner's guide to jets* (preprint)